"SOLUTIONS" ①

Your year in school (semester, degree sought):

33 yrs post Ph.D.*

Your name:

ANDY RUINA

Cornell MAE 4735/5735

Final Exam

Dec 19, 2013

No calculators, books or notes allowed.

5 Problems, 150 minutes, no extra time (Cornell rules)

How to get the highest score?

Please do these things:

? If, when working on a problem, you have any *questions* about what you should or should not assume or write, please read these directions again.

• Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.

$\vec{v}_{\text{vect}}$ Use correct **vector notation**.

A+ Be (I) neat, (II) clear and (III) well organized.

□ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).

>> Make appropriate `Matlab code` clear and correct.
You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
Small syntax errors will have small penalties.

Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.

→| **Justify** your results so a grader can distinguish an informed answer from a guess.

If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).

≈ Work for **partial credit** (from 60 – 100%, depending on the problem)

- Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
- Reduce the problem to a clearly defined set of equations to solve.
- Provide Matlab code which would generate the desired answer (and explain the nature of the output).

Extra sheets. Put your name on each extra sheet, fold it in, and refer to it at the relevant problem.
Note the last page is **blank** for your use. Ask for more extra paper if you need it.

Problem 7:	/25
Problem 8:	/25
Problem 9:	/25
Problem 10:	/25
Problem 11:	/25
Problem 12:	/25

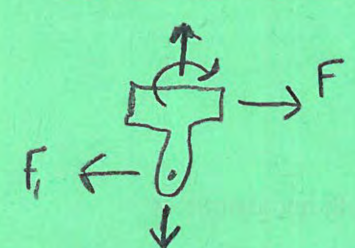
*And still making mistakes

7) 2D. A possibly non-uniform stick with length ℓ, mass m and moment of inertia I about its center of mass G is suspended from a hinge on the stick a distance d from G. The hinge is on a massless trolley (with magnetic frictionless wheels) is forced by a force F and has a horizontal acceleration a_H to the right.

(i) Find the equations of motion assuming a_H is known. That is, find $\ddot{\theta}$ in terms of some or all of $m, I, d, \ell, g, a_H, \theta$ and $\dot{\theta}$ (and *not* F).

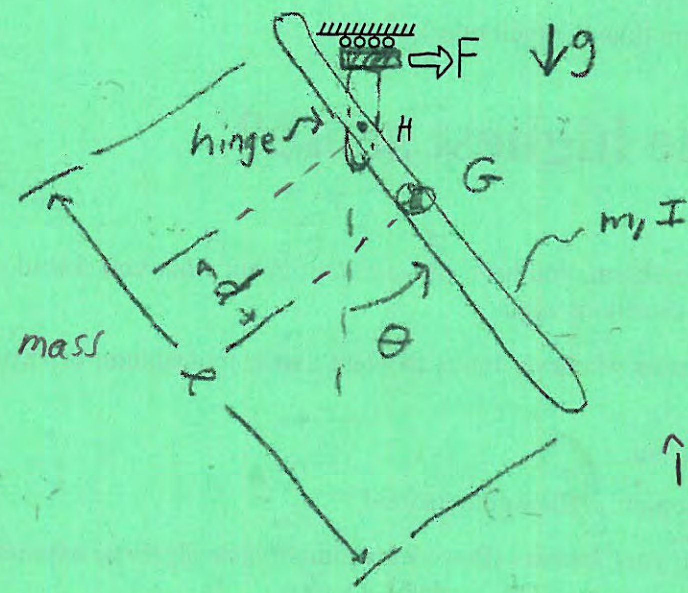
(ii) Assuming $F = 0$, this system has more than one degree of freedom. Assume small θ. Find the normal modes and the angular frequencies of small oscillation in terms of some or all of m, I, d, ℓ and g.

FBD of cart

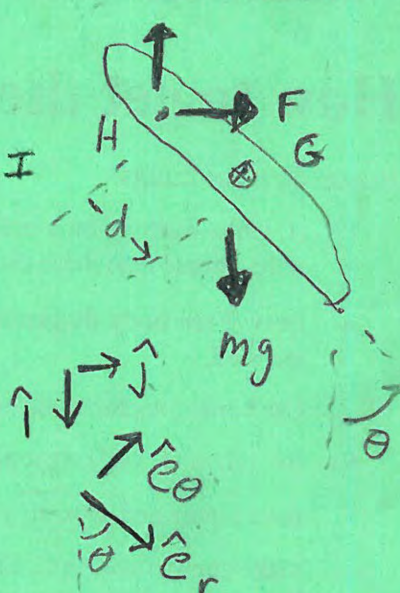


cart has negligible mass

$\Rightarrow F_1 = F$



FBD of stick



Kinematics

$$\vec{a}_G = \vec{a}_H + \vec{a}_{G/H}$$

$$\vec{a}_{G/H} = \ddot{\theta} d\, \hat{e}_\theta - \dot{\theta}^2 d\, \hat{e}_r$$

$$\vec{a}_H = a_H \hat{\jmath}$$

$$\hat{e}_r \times \hat{\jmath} = \cos\theta\, \hat{k}$$

$$\hat{e}_\theta \cdot \hat{\jmath} = \cos\theta$$

$$\hat{e}_r \cdot \hat{\jmath} = \sin\theta$$

AMB/H

$$\sum \vec{M}_{/H} = \dot{\vec{H}}_{/H}$$

$$\Rightarrow -mgd\sin\theta\, \hat{k} = \vec{r}_{G/H} \times m\vec{a}_G + I\ddot{\theta}\hat{k}$$

$\vec{r}_{G/H} = d\hat{e}_r$

given for part i

$$\Rightarrow -mgd\sin\theta\, \hat{k} = (d\hat{e}_r) \times [a_H \hat{\jmath} + \ddot{\theta} d\hat{e}_\theta - \dot{\theta}^2 d\hat{e}_r]m + I\ddot{\theta}\hat{k}$$

$$\{ \quad '' \quad = [d a_H \cos\theta + d^2\ddot{\theta}]m\hat{k} + I\ddot{\theta}\hat{k} \} \; ①$$

(i)

$$\{\ \}\cdot\hat{k} \Rightarrow \boxed{\ddot{\theta} = \frac{-m(gd\sin\theta - d a_H \cos\theta)}{I + md^2} = \frac{-\left(\frac{g}{d}\sin\theta + \frac{a_H}{d}\cos\theta\right)}{1 + I/md^2}}$$

LMB

2 DOF system
$\Leftarrow$ $\Rightarrow$ Need more equations
hence use of LMB

$$\{\sum \vec{F} = m\vec{a}_G\}\cdot\hat{j}$$

$$\Rightarrow \quad F = m\vec{a}_G \cdot \hat{j}$$

$$F = (a_H + \ddot{\theta} d\cos\theta - \dot{\theta}^2 d\sin\theta)m \quad (2)$$

(1) & (2) are 2 eqs for a_H & $\ddot{\theta}$.

Small angle approx. $\Rightarrow$ $\sin\theta \approx \theta$, $\cos\theta \approx 1$, $\dot{\theta}^2 \approx 0$

$$\begin{matrix} (1)\ \&\ (2) \Rightarrow \\ F=0 \Rightarrow \end{matrix} \begin{bmatrix} -mgd\theta = d\,a_H m + (I + md^2)\ddot{\theta} \\ 0 = a_H + \ddot{\theta} d \end{bmatrix} \quad (3)$$

First mode: $\ddot{\theta} = 0 \qquad a_H = 0. \qquad x_H = c_0 + c_1 t$

const vel. motion w/ no rotation

$$\boxed{\omega_1 = 0}$$

Second mode: $-mgd\theta = \underbrace{-\ddot{\theta} d}_{a_H} \cdot dm + (I + md^2)\ddot{\theta}$

$$\Rightarrow \quad I\ddot{\theta} = mgd\theta$$

$$\Rightarrow \begin{bmatrix} \theta =, \text{ say, } \theta_0 \cos\sqrt{\frac{mgd}{I}}\, t \\ x_H = -\theta_0 d \cos\sqrt{\frac{mgd}{I}}\, t \end{bmatrix}$$

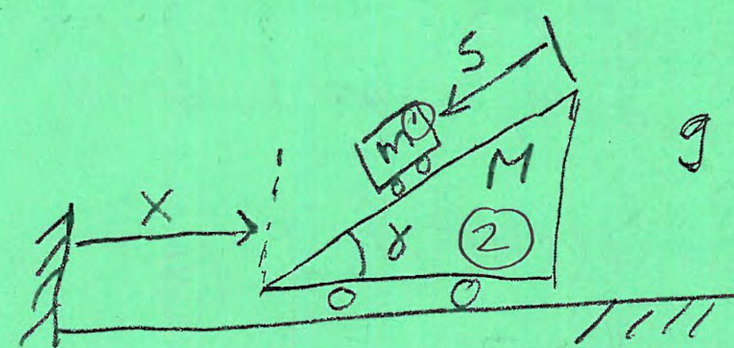
$$\boxed{\omega_2 = \sqrt{\frac{mgd}{I}}} \quad I = I^G$$

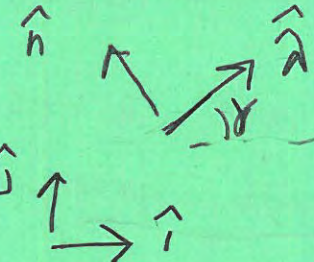
pt. G doesn't move

Note as $I \to 0$ ω / ∞
$\Rightarrow$ nonsense problem for pt. mass.

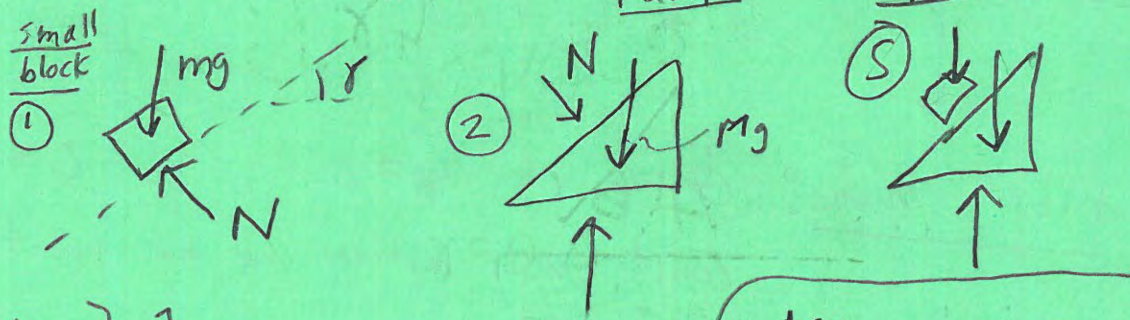
8) A block with mass M slides without friction on a flat level surface. The top surface has slope γ. A smaller block with mass m slides without friction on the sloped top of the lower block. Assume Matlab code has already been written that assigns numerical values to $x, \dot{x}, s, \dot{s}, \gamma, m, M$ and g.

(i) Write Matlab code to find $\ddot{s}$. [*If* you use symbolic commands to generate the equations of motion, you can assume that you have those available as Matlab expressions (That is, I don't expect you to write the commands to convert symbolic expressions into useable matlab). Just, then, clearly explain clearly enough how you would use those expressions. That is, make it clear that you could manage this if a computer was in front of you and you could type 'help' a few times.]





FBDs



{LMB system}$\cdot\hat{\imath}$

$$\left\{\sum \vec{F} = m\vec{a}_1 + M\vec{a}_2\right\}\cdot\hat{\imath}$$

$$0 = \left[m(a_2\hat{\imath} - \ddot{s}\hat{\lambda}) + M a_2\hat{\imath}\right]\cdot\hat{\imath}$$

Kinematics

$\vec{a}_2 = a_2\hat{\imath}$

$\vec{a}_1 = \vec{a}_2 + \vec{a}_{2/1}$

$= a_2\hat{\imath} + \ddot{s}(-\hat{\lambda})$

$\hat{\lambda}\cdot\hat{\imath} = \cos\gamma$

$$\Rightarrow (m+M)a_2 - m\ddot{s}\cos\gamma = 0 \quad (1)$$

{LMB (2)}$\cdot\hat{\lambda}$

$$\left\{-mg\hat{\jmath} = m(a_2\hat{\imath} - \ddot{s}\hat{\lambda})\right\}\cdot\hat{\lambda}$$

$$-\cancel{m}g\sin\gamma = \cancel{m}\cos(\gamma)a_2 - \ddot{s}\cancel{m} \quad (2)$$

$$①,② \Rightarrow \begin{bmatrix} m+M & -m\cos\gamma \\ \cos\gamma & -1 \end{bmatrix} \begin{bmatrix} a_2 \\ \ddot{s} \end{bmatrix} = \begin{bmatrix} 0 \\ -g\sin(\gamma) \end{bmatrix}$$

<u>Matlab</u>

gamma, etc.

$$A = [\; m+M \quad -\cos(\gamma)$$
$$\cos(\gamma) \quad -1 \;];$$

$$b = [\,0\,;\; -\sin\gamma\,];$$

$$X = A \backslash b;$$

$$\boxed{\ddot{s} = X(2);}$$

<u>Solve by hand</u>

$$① \Rightarrow a_2 = \frac{m}{m+M}\cos\gamma\,\ddot{s} \quad ③$$

$$③,② \Rightarrow -g\sin\gamma = \left(\cos^2\gamma\,\frac{m}{m+M} - 1\right)\ddot{s}$$

$$\Rightarrow \boxed{\ddot{s} = \frac{g\sin\gamma}{1-\cos^2\gamma\,\frac{m}{m+M}}}$$

<u>Checks</u>: $M \to \infty \Rightarrow \ddot{s} = g\sin\gamma$ ✓ stationary ramp

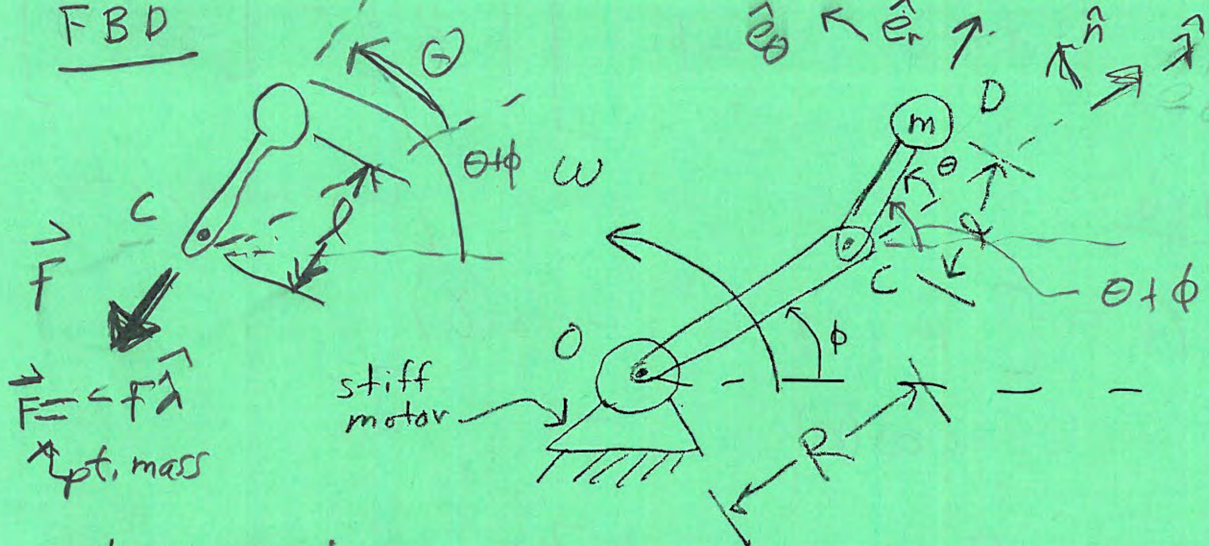
$M \to 0 \Rightarrow \ddot{s} = \frac{g}{\sin\gamma}$ ✓ M is watermellon seed, mass falls down

$\gamma = 0 \Rightarrow \ddot{s} = 0$ ✓

Note: This is almost prob 7i in disguise!

9) Neglect gravity. One link of a pendulum has radius R and is powered by a strong motor to rotate with given $\phi(t)$ (relative to a fixed horizontal line). Thus $\omega = \dot{\phi}$ and $\alpha = \ddot{\phi}$ are known. The second link has length ℓ, is massless, and has a point mass m at the end. Find $\ddot{\theta}$ in terms of some or all of $\theta, \omega, \alpha, R, \ell, m$ and ω.

FBD



Kinematics

$$\vec{a}_D = \vec{a}_C + \vec{a}_{D/C}$$

$$\vec{a}_{D/C} = -\ell(\dot{\theta}+\underbrace{\dot{\phi}}_{\omega})^2\hat{e}_r + \ell(\ddot{\theta}+\underbrace{\ddot{\phi}}_{\alpha})\hat{e}_\theta$$

$$\vec{a}_C = -R\underbrace{\dot{\phi}^2}_{\omega}\hat{\lambda} + R\underbrace{\ddot{\phi}}_{\alpha}\hat{n}$$

$\{LMB\} \cdot \hat{e}_\theta$

$$F\hat{e}_r \cdot \hat{e}_\theta = m\vec{a}_D \cdot \hat{e}_\theta$$

$$\Rightarrow \quad 0 = -R\omega^2\underbrace{\hat{\lambda}\cdot\hat{e}_\theta}_{-\sin\theta} + R\alpha\overbrace{\hat{n}\cdot\hat{e}_\theta}^{\cos\theta} - \ell(\dot{\phi}+\omega)^2\underbrace{\hat{e}_r\cdot\hat{e}_\theta}_{0} + \ell(\ddot{\theta}+\alpha)\underbrace{\hat{e}_\theta\cdot\hat{e}_\theta}_{1}$$

$$\Rightarrow \quad \boxed{\ddot{\theta} = \frac{-R}{\ell}\left[\omega^2\sin\theta + \alpha\cos\theta\right] - \alpha}$$

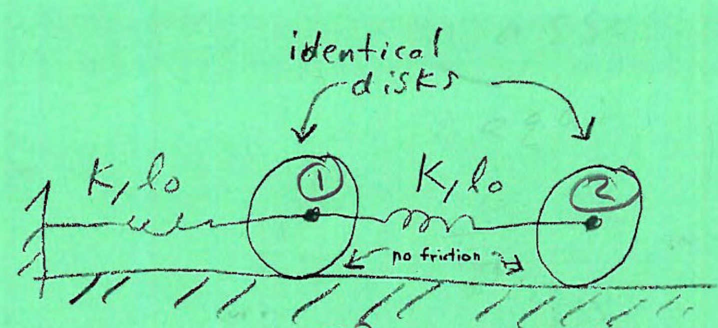
$R\omega^2$ is like g for simple pend.

αR is like horiz. acc. of base.

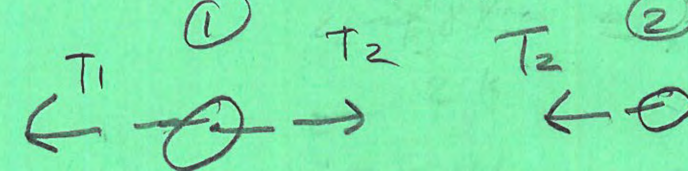
(The final α is because $\ddot{\theta}+\alpha$ is absolute ang. accel.)

There is no page 7. skip to 8.

10) Two round disks with mass m and moment of inertia I (about their centers) slide with no friction on a flat plane. They are connected to each other and to the left wall with two springs, both with stiffness k and rest length ℓ_0. Find one normal mode and it's corresponding frequency.



FBDs



$T_1 = Kx_1$ $\qquad$ $T_2 = K(x_2 - x_1)$

measure from equilibrium

LMB

$$m\ddot{x}_1 = T_2 - T_1 = K(x_2 - x_1) - Kx_1 = Kx_2 - 2Kx_1 \quad \text{A}$$

$$m\ddot{x}_2 = -T_2 = -K(x_2 - x_1) = Kx_1 - Kx_2 \quad \text{B}$$

For normal mode assume

$$x_2 = cx_1$$

A ⇒ $m\ddot{x}_1 = K\underbrace{cx_1}_{x_2} - 2Kx_1 = \overbrace{(c-2)Kx_1}^{-K^*} \quad (1)$

B ⇒ $mc\ddot{x}_1 = Kx_1 - Kcx_1 = (1-c)Kx_1$

$$\Rightarrow m\ddot{x}_1 = \frac{1-c}{c}Kx_1 \quad (2)$$

Eqn 1 must = Eqn 2

$$\Rightarrow \quad c - 2 = \frac{1-c}{c}$$

$$c^2 - 2c = 1 - c$$

$$c^2 - c - 1 = 0$$

$$c = \frac{1 \pm \sqrt{1+4}}{2}$$

mode 1:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$\boxed{\omega_1^2 = \frac{(2-c)k}{m} = \frac{3-\sqrt{5}}{2}\frac{k}{m}}$$

mode 2:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$\boxed{\omega_2^2 = \frac{(2-c)}{m} k = \frac{3+\sqrt{5}}{2}\frac{k}{m}}$$

10

11) Consider the classical 1 DOF spring-mass system with m, k and c.

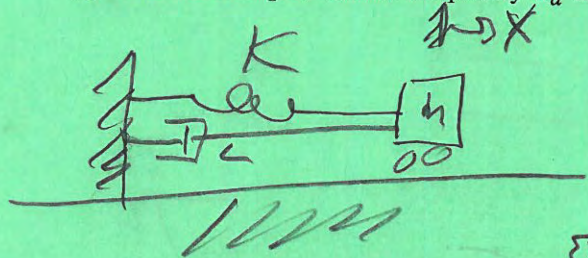
(a) Make a clear picture of the system.

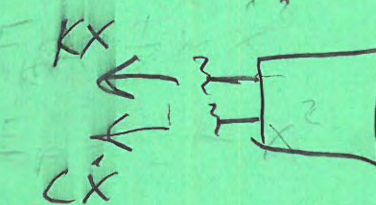
(b) Using any mechanics method you like, find the equations of motion: $m\ddot{x} + c\dot{x} + kx = 0$.

(c) Reduce this to standard form: $\ddot{x} + 2\omega\eta\dot{x} + \omega^2 x = 0$

(d) Define both ω and η with equations. Explain the meaning of both terms with words.

(e) Find the damped natural frequency ω_d in terms of one or both of ω and η.





$$ma = \sum F$$

$\hat{i}\cdot\{\underline{LMB}\} \Rightarrow$ (1 D) $\quad m\ddot{x} = -kx - c\dot{x}$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\boxed{\omega \equiv \sqrt{k/m}}$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \omega^2 x = 0$$

$$\ddot{x} + \frac{c\omega}{m\omega}\dot{x} + \omega^2 x = 0$$

$$\ddot{x} + 2\left(\frac{c}{2m\omega}\right)\omega\dot{x} + \omega^2 x = 0$$

$$\boxed{\eta = \frac{c}{2m\sqrt{k/m}} = \frac{c}{2\sqrt{mk}}}$$

$$\boxed{\ddot{x} + 2\eta\omega x + \omega^2 x = 0}$$ 1

ω = undamped natural frequency

η = amount of damping rel. to critical

11

$\eta < 1 \Rightarrow$ oscillations

$\eta > 1 \Rightarrow$ no oscillations

Assume $\eta < 1$

$$x = e^{\lambda t} \Rightarrow \lambda^2 + 2\eta\omega\lambda + \omega^2 = 0$$

$$\Rightarrow \lambda = \frac{-2\eta\omega \pm \sqrt{4\eta^2\omega^2 - 4\omega^2}}{2}$$

$$= \omega\left(-\eta \pm \sqrt{\eta^2 - 1}\right)$$

$$= \omega\left(-\eta \pm i\sqrt{1-\eta^2}\right)$$

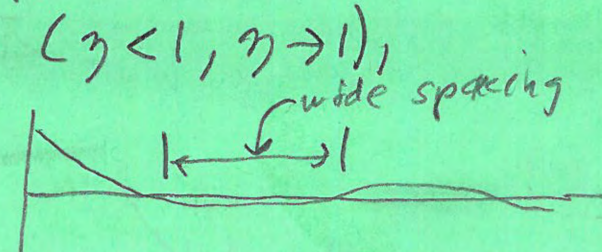
$$\Rightarrow x = Ae^{-\eta\omega t} \underbrace{e^{i\omega\sqrt{(1-\eta^2)}\,t}}_{= \cos(\omega\sqrt{1-\eta^2}\,t) + i\sin(\ )} + B\ldots$$

$$\omega\sqrt{1-\eta^2} = \omega_d$$

$$\Rightarrow \boxed{\omega_d = \omega\sqrt{1-\eta^2} \quad \omega = \sqrt{k/m}, \quad \eta = \frac{c}{2\sqrt{mk}}}$$

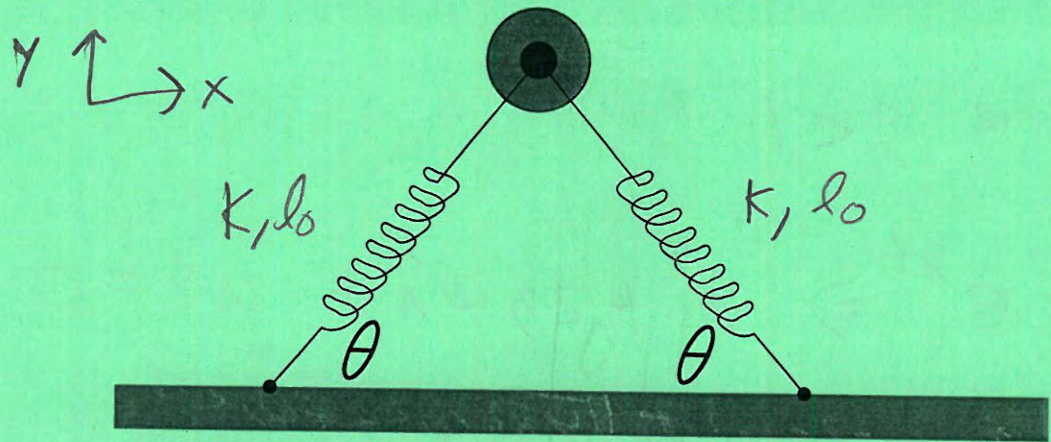
Note: as damping approaches critical from below ($\eta < 1$, $\eta \to 1$),

$\omega_d \to 0$

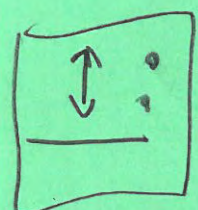


12) Neglect gravity. A mass m is supported by two springs (k, ℓ_0) as shown, each making an angle θ from a fixed horizontal line when in the rest position. Assume small (in the usual sense) motions.

(1) Find the normal modes and their angular frequencies in terms of some or all of m, k, ℓ_0 and θ.



By symmetry the two normal modes are ↕ & ↔,



⇒

$\Delta \ell = y \sin\theta$

$T_1 = k\Delta\ell$

$$\Rightarrow \quad m\ddot{y} = -2\underbrace{T_1}_{k\underbrace{\Delta\ell}_{y\cos\theta}}\sin\theta = -2k\sin^2\theta$$

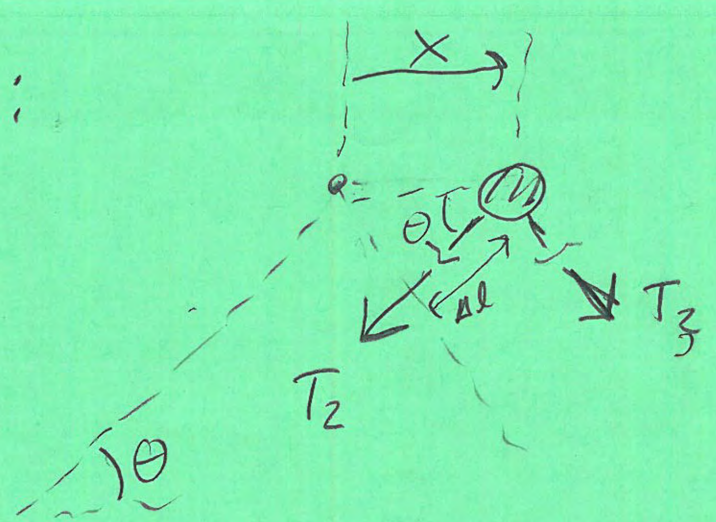
$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{mode shape}} \left[A\sin\left(\underbrace{\sqrt{\frac{2k}{m}}(\sin\theta)}\, t\right) + B\sin(\;) \right]$$

$$\text{freq.} = \boxed{\sqrt{\frac{2k}{m}}\sin\theta = \omega_1}$$

Checks!

$\theta \to \pi/2 \Rightarrow \text{freq} \Rightarrow \sqrt{\frac{2k}{m}}$ ✓, $\theta \to 0 \Rightarrow \text{freq} \to 0$ ✓

$\longleftrightarrow$:



$\Delta l_2 = x\cos\theta$

$T_2 = k\Delta l = kx\cos\theta$

$\Delta l_3 = -x\cos\theta$

$T_3 = -T_2$

Net

$2T_2\cos\theta$

LMB $\Rightarrow$ $\quad m\ddot{x} = -2T_2\cos\theta = -2kx\cos^2\theta$

$$\Rightarrow \boxed{\begin{array}{l} \omega_2 = \sqrt{\dfrac{2k}{m}}\cos\theta \\ \text{mode shape} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}}$$

Checks:

$\theta \to 0, \quad \omega_2 \to \sqrt{\dfrac{2k}{m}}$ ✓

$\theta \to \pi/2, \quad \omega_2 \to 0$ ✓